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On the Representation of Quantum Mechanics on Phase Space

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It is shown that Hilbert-space quantum mechanics can be represented on phase space in the sense that the density operators can be identified with phase-space densities and the observables can be described by functions on phase space. In particular, we consider phase-space representations of quantum mechanics which are related to certain joint position–momentum observables.

1. INTRODUCTION

The problem of reformulating conventional Hilbert-space quantum mechanics in terms of the classical phase space has been investigated by many authors. Wigner (1932), Moyal (1949), and others (e.g., Pool, 1966; Hudson, 1974; O'Connell, 1983) considered a representation of the density operators by density functions on phase space which, however, in general are not nonnegative. Ali and Prugovečki (1977*a*) proved the existence of injective affine maps from the set of all density operators in Hilbert space into the set of all probability measures on phase space [cf. also Srinivas and Wolf (1975)]. Ali and Prugovečki called such mappings *phase-space representations of quantum mechanics*. Furthermore, they showed that the phase-space representations are in bijective correspondence with the so-called *informationally complete* observables and investigated their transformation properties under the Galilei group. Some important aspects of phase-space representations of Hilbert-space quantum mechanics were reviewed and generalized by Guz (1984).

In this paper, we understand phase-space representations in the sense of Ali, Prugovečki, and Guz. In particular, we investigate the description of

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quantum observables by functions on phase space and not only the description of quantum states by probability measures.

Our main result reads as follows. Let a phase-space representation $\hat{T}$ be given which assigns a phase-space density $\rho = \hat{T}W$ to each density operator W. If one fixes a small $\varepsilon > 0$ and arbitrarily (but finitely) many density operators $W_1, W_2, \ldots, W_n$, then for every bounded self-adjoint operator A there exists a real-valued bounded function f on phase space such that

$$\left| \operatorname{tr} W_i A - \int \rho_i(q,p) f(q,p)\, dq\, dp \right| < \varepsilon$$

holds ($\rho_i = \hat{T}W_i$, $i = 1, \ldots, n$). That is, the quantum mechanical expectation values can, in arbitrarily good physical approximation, be calculated as in classical statistical mechanics.

In Section 2, we review some basic concepts of quantum mechanics and fix our notation. In Section 3, we introduce phase-space representations, discuss their relation to informationally complete observables, and prove our main result on the description of observables by functions. The particular case of phase-space representations related to informationally complete joint position–momentum observables is considered in Section 4. We sometimes refer to Singer and Stulpe (1992), where many aspects of phase-space representations are elaborated in greater detail.

2. BASIC CONCEPTS

Conventional Hilbert-space quantum mechanics is based on a complex separable Hilbert space $\mathscr{H}$. We denote the space of all bounded self-adjoint operators in $\mathscr{H}$ by $\mathscr{B}_s(\mathscr{H})$, and the space of all self-adjoint trace-class operators by $\mathscr{T}_s(\mathscr{H})$. As is well known, $\mathscr{B}_s(\mathscr{H})$ can be considered as the dual space $(\mathscr{T}_s(\mathscr{H}))'$ where the duality is given by the trace functional. Let $K(\mathscr{H}) \subset \mathscr{T}_s(\mathscr{H})$ be the convex set of all positive trace-class operators W with $\operatorname{tr} W = 1$, i.e., the set of all density operators, and let $L(\mathscr{H}) \subset \mathscr{B}_s(\mathscr{H})$ be the convex set of all bounded self-adjoint operators A fulfilling $0 \leq A \leq 1$. The density operators describe the *statistical ensembles* of a sort of microsystem which we briefly call *states*. The elements of $L(\mathscr{H})$ describe the *effects*, i.e., the classes of statistically equivalent realistic measurements with the outcomes 0 and 1. For $W \in K(\mathscr{H})$ and $A \in L(\mathscr{H})$, the number $\operatorname{tr} WA \in [0, 1]$ is interpreted to be the *probability for the outcome* 1 *of the effect A in the state W.*

The real Banach space $\mathscr{B}_s(\mathscr{H})$ can be equipped with the weak topology $\sigma(\mathscr{B}_s(\mathscr{H}), \mathscr{T}_s(\mathscr{H}))$ which is the coarsest topology such that all linear functionals given by the elements of $\mathscr{T}_s(\mathscr{H})$ are continuous. We call this topology briefly the *σ-topology*. Since

$$\sigma(\mathscr{B}_s(\mathscr{H}), \mathscr{T}_s(\mathscr{H})) = \sigma(\mathscr{B}_s(\mathscr{H}), K(\mathscr{H}))$$

holds, a neighborhood base of $A \in \mathscr{B}_s(\mathscr{H})$ is given by the sets

$$U(A; W_1, \ldots, W_n; \varepsilon) := \{\tilde{A} \in \mathscr{B}_s(\mathscr{H}) \mid |\text{tr } W_i\tilde{A} - \text{tr } W_iA| < \varepsilon \text{ for } i = 1, \ldots, n\} \tag{1}$$

where $\varepsilon > 0$ and $W_i \in K(\mathscr{H})$. An effect $A \in L(\mathscr{H})$ is physically approximated by $\tilde{A} \in L(\mathscr{H})$ if in many (but finitely many) states $W_1, \ldots, W_n \in K(\mathscr{H})$ the probabilities tr $W_i\tilde{A}$ differ from tr W_iA by an amount less than a small $\varepsilon > 0$. This statement can be tested experimentally and can be characterized mathematically by $\tilde{A} \in U(A; W_1, \ldots, W_n; \varepsilon)$. Hence, the σ-topology, resp. its restriction to $L(\mathscr{H})$, describes the physical approximation of effects (Ludwig, 1983, 1985; Werner, 1983; Haag and Kastler, 1964).

An *observable* F on some measurable space (M, Ξ) is an effect-valued measure on Ξ, i.e., a map $F: \Xi \to L(\mathscr{H})$ satisfying $F(\varnothing) = 0$, $F(M) = 1$, and $F(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} F(B_i)$, where the sets $B_i \in \Xi$ are mutually disjoint and the sum converges in the σ-topology, for instance. Thus, observables are *positive-operator-valued measures* (*POV-measures*), whereas the more common *projection-valued measures* (*PV-measures*) are special cases. A state $W \in K(\mathscr{H})$ and an observable F define a probability measure P_W^F on (M, Ξ) by

$$P_W^F(B) := \text{tr } WF(B) \tag{2}$$

We call P_W^F the *probability distribution of F in the state W*. if F is an observable with real measuring values, i.e., if $(M, \Xi) = (\mathbb{R}, \Xi(\mathbb{R}))$ holds, where $\Xi(\mathbb{R})$ denotes the σ-algebra of Borel sets of $\mathbb{R}$, then the *expectation value of F in the state W* is defined by

$$\langle F \rangle_W := \int \xi P_W^F(d\xi) = \int \text{id}_{\mathbb{R}}\, dP_W^F$$

provided that the integral exists. If $\text{id}_{\mathbb{R}}$ is P_W^F-integrable even for all $W \in K$, then it can be shown (Singer and Stulpe, 1992; Stulpe, 1986, 1988) that the integral $\int \text{id}_{\mathbb{R}}\, dF =: A \in \mathscr{B}_s(\mathscr{H})$ exists in a σ-weak sense. In consequence, we obtain

$$\langle F \rangle_W = \int \text{id}_{\mathbb{R}}\, d(\text{tr } WF(\cdot)) = \text{tr}\left(W \int \text{id}_{\mathbb{R}}\, dF\right)$$

resp.

$$\langle F \rangle_W = \text{tr } WA \tag{3}$$

Hence, for any $W \in K(\mathscr{H})$and any $A \in \mathscr{B}_s(\mathscr{H})$, one can interpret the real number tr WA as the expectation value of some observable. According to (1) and (3), the σ-topology then describes the physical approximation of observables.

Adopting a terminology introduced by Ali and Prugovečki (1977*a*,*b*; Prugovečki, 1977), we call a family $\{F_\alpha\}_{\alpha \in I}$ of observables on (M_α, Ξ_α) *informationally complete* if every state is determined by the probability distributions (2) of all F_α, i.e., if for any two states W_1, $W_2 \in K(\mathscr{H})$,

$$P_{W_1}^{F_\alpha} = P_{W_2}^{F_\alpha} \qquad \text{for all} \quad \alpha \in I$$

implies $W_1 = W_2$. It is remarkable that especially one observable can be informationally complete. Physically interesting examples for such observables are given in Ali and Prugovečki (1977*a*,*b*); we shall consider these examples in Section 4. Some mathematical aspects of informationally completeness are discussed in Singer and Stulpe (1992). In particular, it is shown that the existence of single informationally complete observables can be concluded from the norm-separability of $\mathscr{H}$, resp., $\mathscr{T}_S(\mathscr{H})$. Moreover, informationally complete POV-measures cannot be PV-measures.

3. PHASE-SPACE REPRESENTATIONS

Let a set $\Omega \neq \varnothing$ and a σ-algebra Σ in Ω be given, i.e., a nontrivial measurable space (Ω, Σ). We call (Ω, Σ) *phase space*. Denote the space of all σ-additive real-valued measures on Σ by $\mathscr{M}_{\mathbb{R}}(\Omega, \Sigma)$ and the convex set of all probability measures by $K(\Omega, \Sigma)$. By means of $\|\nu\| := |\nu|(\Omega)$, where $|\nu|$ is the total variation of $\nu \in \mathscr{M}_{\mathbb{R}}(\Omega, \Sigma)$, $\mathscr{M}_{\mathbb{R}}(\Omega, \Sigma)$ becomes a real Banach space. Now let $\lambda \neq 0$ be a fixed σ-finite (not necessarily finite) positive measure on Σ, i.e., $(\Omega, \Sigma, \lambda)$ is a nontrivial σ-finite measure space. In the standard case, λ is the Lebesgue measure defined on the σ-algebra of Borel sets of the usual phase space. The real Banach space $\mathscr{L}^1_{\mathbb{R}}(\Omega, \Sigma, \lambda)$ can be considered as a closed subspace of $\mathscr{M}_{\mathbb{R}}(\Omega, \Sigma)$. Furthermore,

$$(\mathscr{L}^1_{\mathbb{R}}(\Omega, \Sigma, \lambda))' = \mathscr{L}^\infty_{\mathbb{R}}(\Omega, \Sigma, \lambda)$$

holds. By $K(\Omega, \Sigma, \lambda)$ we denote the convex set of all phase-space densities, i.e., the set of all probability densities $\rho \in \mathscr{L}^1_{\mathbb{R}}(\Omega, \Sigma, \lambda)$ defined by $\rho(\omega) \geq 0$ for λ-almost all $\omega \in \Omega$ and $\int \rho \, d\lambda = 1$.

A *phase-space representation of quantum mechanics* is an affine map that assigns to every state $W \in K(\mathscr{H})$ injectively a phase-space density $\rho \in K(\Omega, \Sigma, \lambda)$ or, more generally, a probability measure $\mu \in K(\Omega, \Sigma)$. It is easy to show that the injective affine maps $\tilde{T}: K(\mathscr{H}) \to K(\Omega, \Sigma)$ correspond bijectively with the injective positive linear maps $T: \mathscr{T}_s(\mathscr{H}) \to \mathscr{M}_{\mathbb{R}}(\Omega, \Sigma)$ having the property $TK(\mathscr{H}) \subseteq K(\Omega, \Sigma)$.

Definition 3.1. We call a linear map $T:\mathcal{T}_s(\mathcal{H})\to\mathcal{M}_{\mathbb{R}}(\Omega,\Sigma)$ a *phase-space representation of quantum mechanics on* (Ω,Σ) if (i) $TK(\mathcal{H})\subseteq K(\Omega,\Sigma)$, (ii) T is injective. If (Ω,Σ,λ) is a σ-finite measure space, then an injective linear map $\hat{T}:\mathcal{T}_s(\mathcal{H})\to\mathcal{L}^1_{\mathbb{R}}(\Omega,\Sigma,\lambda)$ with $\hat{T}K(\mathcal{H})\subseteq K(\Omega,\Sigma,\lambda)$ is called a *phase-space representation of quantum mechanics on* (Ω,Σ,λ).

Condition (i) implies that T is a positive bounded linear map with $\|T\|=1$. The same holds for $\hat{T}$. It is a remarkable fact that injective affine maps from $K(\mathcal{H})$ into $K(\Omega,\Sigma)$ do exist. In fact, every informationally complete observable gives rise to a phase-space representation.

Proposition 3.2. Every informationally complete observable F on $(M,\Xi):=(\Omega,\Sigma)$ defines a phase-space representation T on (Ω,Σ) by

$$(TV)(A):=\operatorname{tr} VF(A) \tag{4}$$

where $V\in\mathcal{T}_s(\mathcal{H})$ and $A\in\Sigma$. In particular, for $W\in K(\mathcal{H})$ we have

$$TW=\operatorname{tr} WF(\cdot)=P^F_W$$

Proof. Obviously, $TV\in\mathcal{M}_{\mathbb{R}}(\Omega,\Sigma)$, $TW=P^F_W\in K(\Omega,\Sigma)$, and the map T is linear. Since F is informationally complete, T is injective on $K(\mathcal{H})$. Let V_1 and V_2 be positive trace-class operators and assume

$$TV_1=TV_2 \tag{5}$$

Then it follows from (4) by setting $A=\Omega$ that $\operatorname{tr} V_1=\operatorname{tr} V_2=:\alpha$. For $\alpha=0$, we obtain $V_1=V_2=0$. For $\alpha\neq 0$, divide (5) by α and observe that $(1/\alpha)V_1$ and $(1/\alpha)V_2$ are density operators. Consequently, $V_1=V_2$ holds. Finally, assume (5) for arbitrary V_1, $V_2\in\mathcal{T}_s(\mathcal{H})$. Decomposing V_1 and V_2 into positive operators, we obtain

$$T(V_1^+-V_1^-)=T(V_2^+-V_2^-)$$

resp.

$$T(V_1^++V_2^-)=T(V_2^++V_1^-)$$

and conclude $V_1^++V_2^-=V_2^++V_1^-$, resp. $V_1=V_2$. Hence, T is injective. ■

Conversely, one can prove (Singer and Stulpe, 1992) that every phase-space representation T on (Ω,Σ) determines uniquely an informationally complete observable F such that $TV=\operatorname{tr} VF(\cdot)$ is satisfied.

Let τ be the canonical embedding of $\mathcal{L}^1_{\mathbb{R}}(\Omega,\Sigma,\lambda)$ into $\mathcal{M}_{\mathbb{R}}(\Omega,\Sigma)$, i.e., $(\tau\rho)(A):=\int_A \rho\, d\lambda$, where $\rho\in\mathcal{L}^1_{\mathbb{R}}(\Omega,\Sigma,\lambda)$ and $A\in\Sigma$. The map τ and its inverse τ^{-1} defined on $\tau\mathcal{L}^1_{\mathbb{R}}(\Omega,\Sigma,\lambda)$ are linear, isometric, and positive. Every phase-space representation $\hat{T}$ on (Ω,Σ,λ) defines a phase-space representation T on (Ω,Σ) by $T:=\tau\hat{T}$. Conversely, if T is a phase-space representation on (Ω, Σ) such that all measures TV are absolutely continuous with respect to λ, then a map $\hat{T}$ can be defined by $\hat{T} := \tau^{-1}T$. In the next section, we shall discuss a class of physically interesting examples for phase-space representations of the form $T = \tau\hat{T}$.

We remark that an injective affine map $\tilde{T}: K(\mathscr{H}) \to K(\Omega, \Sigma)$ and in this sense a phase-space representation T on (Ω, Σ) cannot be bijective (Singer and Stulpe, 1992). The same statement holds for injective affine maps from $K(\mathscr{H})$ into $K(\Omega, \Sigma, \lambda)$ and phase-space representations on $(\Omega, \Sigma, \lambda)$.

By means of a phase-space representation of quantum mechanics, the states $W \in K(\mathscr{H})$ can be identified with probability measures. Our main result concerns the corresponding description of the observables $A \in \mathscr{B}_s(\mathscr{H})$ and is formulated as the subsequent theorem. To prove it, we need the following lemma.

Lemma 3.3. Let $\mathscr{V}_1$ and $\mathscr{V}_2$ be Banach spaces with duals $\mathscr{V}_1'$ and $\mathscr{V}_2'$, $T: \mathscr{V}_1 \to \mathscr{V}_2$ a bounded linear map, and $T': \mathscr{V}_2' \to \mathscr{V}_1'$ the adjoint map. If T is injective, then the range $R(T') := T'\mathscr{V}_2'$ is a subspace being $\sigma(\mathscr{V}_1', \mathscr{V}_1)$-dense in $\mathscr{V}_1'$.

Proof. Assume

$$\overline{R(T')}^{\sigma(\mathscr{V}_1', \mathscr{V}_1)} \neq \mathscr{V}_1'$$

Then, according to a consequence of the Hahn–Banach theorem, there exists a $\sigma(\mathscr{V}_1', \mathscr{V}_1)$-continuous linear functional $\Lambda \neq 0$ on $\mathscr{V}_1'$ such that

$$\Lambda(l) = 0 \qquad \text{for all} \quad l \in \overline{R(T')}^{\sigma(\mathscr{V}_1', \mathscr{V}_1)}$$

Since the $\sigma(\mathscr{V}_1', \mathscr{V}_1)$-continuous linear functionals on $\mathscr{V}_1'$ are just the ones that are represented by the elements of $\mathscr{V}_1$, we have

$$\Lambda(l) = l(v) = 0$$

for all $l \in R(T')$, where $v \in \mathscr{V}_1$, $v \neq 0$, corresponds to Λ. From this it follows that

$$(T'\tilde{l})(v) = \tilde{l}(Tv) = 0$$

for all $\tilde{l} \in \mathscr{V}_2'$ and some $v \neq 0$. Consequently, we obtain $Tv = 0$ for some $v \neq 0$. This is a contradiction because T is presupposed to be injective. Hence, $R(T')$ is $\sigma(\mathscr{V}_1', \mathscr{V}_1)$-dense in $\mathscr{V}_1'$. ∎

Since the dual space of $\mathscr{M}_{\mathbb{R}}(\Omega, \Sigma)$ is a very abstract object, we formulate the following theorem only for phase-space representations on $(\Omega, \Sigma, \lambda)$. In Singer and Stulpe (1992) a corresponding theorem for phase-space representations on (Ω, Σ) is presented where the space $(\mathscr{M}_{\mathbb{R}}(\Omega, \Sigma))'$ is replaced by the subspace $\mathscr{F}_{\mathbb{R}}(\Omega, \Sigma)$ of all real-valued, bounded, Σ-measurable functions on Ω. [Remember that, in contrast to $\mathscr{F}_{\mathbb{R}}(\Omega, \Sigma)$, the elements of $\mathscr{L}^{\infty}_{\mathbb{R}}(\Omega, \Sigma, \lambda)$ are classes of λ-essentially bounded functions.]

Theorem 3.4. Let a σ-finite measure space $(\Omega, \Sigma, \lambda)$ and a phase-space representation $\hat{T}: \mathscr{T}_s(\mathscr{H}) \to \mathscr{L}^1_{\mathbb{R}}(\Omega, \Sigma, \lambda)$ be given. Then, for every $A \in \mathscr{B}_s(\mathscr{H})$, every $\varepsilon > 0$, and any finitely many states $W_1, W_2, \ldots, W_n \in K(\mathscr{H})$, there exists a function $f \in \mathscr{L}^{\infty}_{\mathbb{R}}(\Omega, \Sigma, \lambda)$ such that

$$\left| \operatorname{tr} W_i A - \int \rho_i f \, d\lambda \right| < \varepsilon$$

holds, where $\rho_i := \hat{T} W_i$ $(i = 1, \ldots, n)$.

Proof. According to the preceding lemma, $R(\hat{T}') = \hat{T}' \mathscr{L}^{\infty}_{\mathbb{R}}(\Omega, \Sigma, \lambda)$ is σ-dense in $\mathscr{B}_s(\mathscr{H})$. This implies that, for any $A \in \mathscr{B}_s(\mathscr{H})$, every neighborhood $U(A; W_1, \ldots, W_n; \varepsilon)$ of the form (1) contains an element $\hat{T}'f \in R(\hat{T}')$. Thus, we obtain

$$|\operatorname{tr} W_i A - \operatorname{tr} W_i(\hat{T}'f)| < \varepsilon$$

for every $\varepsilon > 0$ and any $W_1, \ldots, W_n \in K(\mathscr{H})$. Now, the assertion follows from

$$\operatorname{tr} W_i(\hat{T}'f) = \int (\hat{T} W_i) f \, d\lambda = \int \rho_i f \, d\lambda \quad \blacksquare$$

Given a phase-space representation of quantum mechanics on $(\Omega, \Sigma, \lambda)$, the states $W \in K(\mathscr{H})$ can be characterized by phase-space densities. The theorem states that, moreover, the observables $A \in \mathscr{B}_s(\mathscr{H})$ can, in arbitrarily good physical approximation, be described by functions on phase space such that the quantum mechanical expectation values can be calculated as integrals. This situation is fairly close to classical statistical mechanics. In particular, one can work with the same small $\varepsilon > 0$ and the same many states $W_1, \ldots, W_n$ for all observables.

An operator $A \in L(\mathscr{H})$ can be interpreted as an effect. According to the theorem, the effects $A \in L(\mathscr{H})$ can be represented by functions $f \in \mathscr{L}^{\infty}_{\mathbb{R}}(\Omega, \Sigma, \lambda)$ such that the probabilities $\operatorname{tr} W_i A$ coincide approximately with $\int \rho_i f \, d\lambda$. However, it does not follow that $f \in L(\Omega, \Sigma, \lambda)$ holds, i.e., a function f representing an effect need not satisfy $0 \leq f \leq 1$ λ-a.e.

4. JOINT POSITION–MOMENTUM OBSERVABLES

We now discuss phase-space representations of quantum mechanics that are related to informationally complete joint position–momentum observables. For simplicity, we consider spinless particles moving in one spatial dimension.

We are working with the Hilbert space $\mathscr{H}:=\mathscr{L}^2_{\mathbb{C}}(\mathbb{R}, \Xi(\mathbb{R}), \lambda)$, where $\Xi(\mathbb{R})$ is the σ-algebra of Borel sets of $\mathbb{R}$ and λ the Lebesgue measure. Let $u\in\mathscr{H}$ be a bounded function of norm 1 and define

$$u_{qp}(x):=e^{ipx}u(x-q)$$

for $q, p\in\mathbb{R}$. It is well known that the family $\{u_{qp}\}_{(q,p)\in\mathbb{R}^2}$ is a continuous resolution of the identity in $\mathscr{H}$, i.e.,

$$1=\frac{1}{2\pi}\int |u_{qp}\rangle\langle u_{qp}|\, dq\, dp$$

holds where the integral exists in the weak sense [for a rigorous proof, see Davies (1976)]. In consequence, a POV-measure F on the phase space $(\Omega, \Sigma):=(\mathbb{R}^2, \Xi(\mathbb{R}^2))$ is defined by

$$F(A):=\frac{1}{2\pi}\int_A |u_{qp}\rangle\langle u_{qp}|\, dq\, dp \tag{6}$$

where $A\in\Xi(\mathbb{R}^2)$ is a Borel set. As we shall see, the two-dimensional Lebesgue measure λ^2, $dq\, dp:=\lambda^2(d(q,p))$, corresponds to the general measure λ of the previous section. We call F a *joint position–momentum observable*. The marginal observables are given by

$$\begin{aligned} F^Q(B)&:=F(B\times\mathbb{R})=\int \chi_B * |u|^2\, dE^Q \\ F^P(B)&:=F(\mathbb{R}\times B)=\int \chi_B * |\hat{u}|^2\, dE^P \end{aligned} \tag{7}$$

where $B\in\Xi(\mathbb{R})$, χ_B is the characteristic function of B, E^Q and E^P are the spectral measures of the position and momentum operator, and $\hat{u}$ denotes the Fourier transform of u (Davies, 1976). That is, F^Q is an *approximate position observable* and F^P an *approximate momentum observable* (Stulpe *et al.*, 1988). The integrals in (7) are understood in the weak or σ-weak sense. Finally, the definition (6) for joint position–momentum observables as well as the following considerations can be slightly generalized. Namely, one can replace the function u by some density operator in $\mathscr{H}$ and the configuration space $\mathbb{R}$ by $\mathbb{R}^N$.

Ali and Prugovečki (1977b) proved that the observable given by (6) is informationally complete if and only if the Weyl transform

$$(q,p)\mapsto \operatorname{tr} P_u U_{qp}=\langle u|u_{qp}\rangle$$

($P_u:=|u\rangle\langle u|$, $U_{qp}:=e^{ipQ}\, e^{-iqP}$) is different from zero for λ^2-almost all $(q,p)\in\mathbb{R}^2$. In particular, there exist informationally complete joint position–

momentum obervables F. In contrast to the informational completeness of F, neither the pair of the PV-measures E^Q and E^P (e.g., Prugovečki, 1977; Stulpe and Singer, 1990) nor the pair of the marginal observables F^Q and F^P (Ali and Prugovečki, 1977*a*) is informationally complete.

Let F be an informationally complete observable according to (6) and T the corresponding phase-space representation on $(\mathbb{R}^2, \Xi(\mathbb{R}^2))$ according to Proposition 3.2. For every density operator $W \in K(\mathcal{H})$, the assigned probability measure $TW = P_W^F$ has the continuous phase-space density $(q, p) \mapsto \rho(q, p) := (1/2\pi)\langle u_{qp} | W u_{qp}\rangle$. This implies that a phase-space representation $\hat{T}$ on $(\mathbb{R}^2, \Xi(\mathbb{R}^2), \lambda^2)$ is determined by $\hat{T}W := \rho$. We next calculate a representation for the adjoint map $\hat{T}'$. For all $W \in K(\mathcal{H})$ and all $f \in \mathcal{L}_{\mathbb{R}}^{\infty}(\mathbb{R}^2, \Xi(\mathbb{R}^2), \lambda^2)$, we have

$$\begin{aligned} \operatorname{tr} W(\hat{T}'f) &= \int (\hat{T}W) f \, d\lambda^2 = \int \rho(q, p) f(q, p) \, dq \, dp \\ &= \int f \, dP_W^F = \int f \, d(\operatorname{tr} WF(\cdot)) = \operatorname{tr}\left(W \int f \, dF\right) \end{aligned}$$

where $\int f \, dF$ is understood in the σ-weak sense (Singer and Stulpe, 1992; Stulpe, 1986, 1988) and does not depend on the representative of the class of λ^2-essentially bounded functions. Hence, we obtain

$$\hat{T}'f = \int f \, dF$$

and in consequence

$$\hat{T}'f = \frac{1}{2\pi} \int f(q, p) |u_{qp}\rangle \langle u_{qp}| \, dq \, dp \tag{8}$$

According to Lemma 3.3, every operator $A \in \mathcal{B}_s(\mathcal{H})$ can be approximated with respect to the σ-topology by some $\hat{T}'f \in \mathcal{B}_s(\mathcal{H})$. In particular, the spectral projections $E^Q(B)$ and $E^P(B)$ can be approximated physically arbitrarily well by operators of the form (8). Roughly speaking, with suitable functions f_B^Q and f_B^P for every $B \in \Xi(\mathbb{R})$ we have

$$\begin{aligned} E^Q(B) &\approx \frac{1}{2\pi} \int f_B^Q(q, p) |u_{qp}\rangle \langle u_{qp}| \, dq \, dp \\ E^P(B) &\approx \frac{1}{2\pi} \int f_B^P(q, p) |u_{qp}\rangle \langle u_{qp}| \, dq \, dp \end{aligned} \tag{9}$$

A similar approximation of $E^Q(B)$ and $E^P(B)$ is given by equations (7):

$$\begin{aligned} F^Q(B) &= \frac{1}{2\pi} \int \chi_{B \times \mathbb{R}}(q, p) |u_{qp}\rangle \langle u_{qp}| \, dq \, dp \\ &= \int \chi_B * |u|^2 \, dE^Q \approx E^Q(B) \\ F^P(B) &= \frac{1}{2\pi} \int \chi_{\mathbb{R} \times B}(q, p) |u_{qp}\rangle \langle u_{qp}| \, dq \, dp \\ &= \int \chi_B * |\hat{u}|^2 \, dE^P \approx E^P(B) \end{aligned} \tag{10}$$

However, it is intuitively clear by the properties of the Fourier transformation that, if one of the approximations (10) is good, the other one is bad. In contrast, both approximations (9) may be good. Hence, in a certain sense, the observables of position and momentum both can be approximated by functions of the joint position–momentum observable F arbitrarily well.

We conclude these considerations with a precise statement. Fix a small $\varepsilon > 0$ and many density operators $W_1, \ldots, W_n$. Then, by Theorem 3.4, for every operator $A \in \mathscr{B}_s(\mathscr{H})$ there exists a function $f \in \mathscr{L}^\infty_\mathbb{R}(\mathbb{R}^2, \Xi(\mathbb{R}^2), \lambda^2)$ such that

$$\left| \operatorname{tr} W_i A - \int \rho_i(q, p) f(q, p) \, dq \, dp \right| < \varepsilon$$

holds, where the functions $(q, p) \mapsto \rho_i(q, p) = (1/2\pi)\langle u_{qp} | W_i u_{qp}\rangle$ are continuous phase-space densities.

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